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A Theory of Generally Invariant Lagrangians for the Metric Fields. I.

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The second-order generally invariant Lagrangians for the metric fields are studied within the framework of the Ehresmann theory of jets. Such a Lagrangian is a function on an appropriate fiber bundle whose structure group is the group L_n^3 of invertible 3-jets with source and target at the origin 0 of the real, n-dimensional Euclidean space R^n, and whose type fiber is the manifold $T_n^2(R^{n*} \odot R^{n*})$ of 2-jets with source at $0 \in R^n$ and target in the symmetric tensor product $R^{n*} \odot R^{n*}$. Explicit formulas for the action of L_n^3 on $T_n^2(R^{n*} \odot R^{n*})$ are considered, and a complete system of differential identities for the generally invariant Lagrangians is obtained.

1. INTRODUCTION

As the geometric structure of the generally invariant variational problems in fiber bundles is satisfactorily understood (Krupka and Trautman, 1974), further interest in these variational problems is shifted to the existential and computational aspects of the theory. A natural question is posed as to what are all possible generally invariant Lagrangians for a tensor bundle of a given type.

The general theory tells us that each generally invariant Lagrangian is defined on the type fiber of a fiber bundle, and is invariant under a Lie transformation group acting on this type fiber. Let Ξ_ι, $\iota = 1, 2, \ldots, k$, be any fundamental vector fields generating this Lie transformation group. Then if a function L on the type fiber defines a generally invariant Lagrangian, it must satisfy the system

$$\Xi_\iota(L) = 0 \tag{1.1}$$

of linear, homogeneous, first-order partial differential equations. This suggests that we should look for the generally invariant functions among the solutions of the system (1.1).

Recently, a method for determining the Lie transformation groups in a very general case of the r-jet prolongation of fiber bundles, associated with the bundle of frames, has been proposed (Krupka, 1974). This method is based on the theory of jet prolongations of principal fiber bundles (Kolář, 1971a, 1971b).

In this paper we study the second-order generally invariant Lagrangians for the bundle of metrics. It is the merit of the relativity theory that the Lagrangians of this kind have been studied in literature to a great extent. This provides us a possibility to complete the well-known results by a deeper, rigorous geometric insight as well as to illustrate how the general method works in a concrete, not so simple situation.

More precisely, let X be an n-dimensional, real differential manifold and E the bundle of second-order, symmetric covariant tensors on X. Denote by R^n the real, n-dimensional, Euclidean vector space, R^{n*} its dual vector space. Obviously, E has the symmetric tensor product $R^{n*} \odot R^{n*}$ for its type fiber. The second-jet prolongation $\mathscr{J}^2E$ of E, the domain of relativistic Lagrangians for the metric fields on X, has the type fiber $T_n^2(R^{n*} \odot R^{n*})$, the manifold of 2-jets of maps from R^n to $R^{n*} \odot R^{n*}$ with source at $0 \in R^n$. Let L_n^3 be the group of invertible 3-jets with source and target at $0 \in R^n$. Our main result consists in the formulation of the system (1.1) of the identities relative to the natural action of L_n^3 on $T_n^2(R^{n*} \odot R^{n*})$. Further results on the theory of the second-order generally invariant Lagrangians for the metric fields may be found in the second part of this work (Krupka, 1976).

Throughout, the standard notation of the differential calculus, manifolds, and the theory of jets is used (see, e.g., Dieudonné, 1969, 1972; Ehresmann, 1953). To perform numerous differentiations precisely, we use both symbols D_i and $\partial/\partial x_i$ for the partial-derivative operators. The Einstein summation convention is used.

2. PRELIMINARIES

In this section we briefly recall some facts concerning the Lie transformation groups (Dieudonné, 1972), vector-field systems (Hermann, 1968), and the theory of generally invariant Lagrangians in tensor bundles (Krupka, 1974).

Let G be a Lie group acting on a manifold X by the map $G \times X \ni (g, x) \to gx \in X$. Each $x \in X$ gives rise to the map

$$G \ni g \to \tilde{x}(g) = gx \in X \tag{2.1}$$

By means of this map one can construct a homomorphism l from the Lie algebra l_G of G onto a Lie algebra $l_G(X)$ of some vector fields on X. This is defined as follows. For $x \in X$ and $\xi \in l_G$ we put

$$l(\xi)(x) = T_e\tilde{x}\xi \tag{2.2}$$

where e is the identity element of G, l_G is identified with T_eG, and $T_e\tilde{x}$ denotes the tangent map of $\tilde{x}$ at e. In this way the Lie algebra $l_G(X)$ can be completely described by the image of any basis of l_G.

Recall that a real function L, defined on an open, G-invariant subset U of X, is said to be G-invariant if

$$L(gx) = L(x) \tag{2.3}$$

for all $x \in U$ and all $g \in G$. Denote by Ξ_ι, $\iota = 1, 2, \ldots, k$, any subset of $l_G(X)$ generating the whole Lie algebra $l_G(X)$. Obviously, if L is G-invariant then (1.1) holds for all ι. This means that all G-invariant functions can be found among the solutions of this system of equations.

The above considerations can be applied to the natural action of the group L_n^3 on the manifold $T_n^2(R^{n*} \odot R^{n*})$ (Krupka, 1974). Let $(a, g) \to ag$ denote the natural action of L_n^1 $(= GL_n(R))$ on $R^{n*} \odot R^{n*}$. Within the context of the theory of jet bundles, choose a point $Q = j_0^2 f \in T_n^2(R^{n*} \odot R^{n*})$. Then the map (2.1) is of the form

$$L_n^3 \ni j_0^3\alpha \to \tilde{Q}(j_0^3\alpha) = j_0^2((D\alpha\circ\alpha^{-1})f\alpha^{-1}) \in T_n^2(R^{n*} \odot R^{n*}) \tag{2.4}$$

or, more precisely, $\tilde{Q}(j_0^3\alpha) = j_0^2(\tilde{\alpha}f\alpha^{-1})$, where $\tilde{\alpha} = D\alpha\circ\alpha^{-1}$ maps a neighborhood of $0 \in R^n$ to L_n^1, $\tilde{\alpha}(x) = j_0^1(t_x\alpha t_{-\alpha^{-1}(x)})$, and t_x denotes the translation of R^n sending the point $x \in R^n$ to the origin $0 \in R^n$. The homomorphism $l\colon l_{L_n^3} \to l_{L_n^3}(T_n^2(R^{n*} \odot R^{n*}))$ of Lie algebras is then constructed by (2.2).

3. IDENTITIES FOR THE SECOND-ORDER GENERALLY INVARIANT LAGRANGIANS

In this section we establish the equations (1.1) for the action (2.4) of L_n^3 on $T_n^2(R^{n*} \odot R^{n*})$.

In order to discuss the structure of the map (2.4) in detail, we need some coordinates on the considered spaces.

Let us consider the group L_n^1. On L_n^1 there exist global coordinates a_i^p, $1 \leqslant i, p \leqslant n$, defined by

$$a_i^p(j_0^1\alpha) = D_i\alpha_p^{-1}(0)$$

Here, α_p^{-1} denotes the pth component of the map α^{-1}. We shall write for short $j_0^1\alpha = (a_i^p)$. Similarly, on L_n^3 there exist global coordinates a_i^p, a_{ij}^p, a_{ijk}^p, $1 \leqslant p \leqslant n$, $1 \leqslant i \leqslant j \leqslant k \leqslant n$, defined by

$$\begin{aligned} a_i^p(j_0^3\alpha) &= D_i\alpha_p^{-1}(0) \\ a_{ij}^p(j_0^3\alpha) &= D_iD_j\alpha_p^{-1}(0) \\ a_{ijk}^p(j_0^3\alpha) &= D_iD_jD_k\alpha_p^{-1}(0) \end{aligned} \tag{3.1}$$

We shall write $j_0^3\alpha = (a_i^p, a_{ij}^p, a_{ijk}^p)$.

Next, let us consider the space $R^{n*} \odot R^{n*}$. Let e_i be the natural basis of the vector space R^n, e^i the dual basis of R^{n*}. Each element $g \in R^{n*} \odot R^{n*}$ is uniquely written in the form

$$g = g_{ij}e^i \otimes e^j$$

where $g_{ij} = g_{ji}$. We take the numbers g_{ij}, $1 \leqslant i \leqslant j \leqslant n$, for the coordinates on $R^{n*} \odot R^{n*}$ and write $g_{ij} = g_{ij}(g)$ or just $g = (g_{ij})$. Similarly, let $Q \in T_n^2(R^{n*} \odot R^{n*})$, and choose a map f of a neighborhood of $0 \in R^n$ into $R^{n*} \odot R^{n*}$ such that $Q = j_0^2f$. Then to each point x from a neighborhood of $0 \in R^n$ we are given an element of $R^{n*} \odot R^{n*}$,

$$f(x) = g_{ij}(f(x))e^i \otimes e^j$$

A system of coordinates on $T_n^2(R^{n*} \odot R^{n*})$ is defined by

$$\begin{aligned} g_{ij}(Q) &= g_{ij}(f(0)) \\ g_{ij,k}(Q) &= D_k(g_{ij}f)(0) \\ g_{ij,kl}(Q) &= D_kD_l(g_{ij}f)(0) \end{aligned} \tag{3.2}$$

where $1 \leqslant i \leqslant j \leqslant n$, $1 \leqslant k \leqslant l \leqslant n$. We write $j_0^2f = (g_{ij}, g_{ij,k}, g_{ij,kl})$.

The standard left action of L_n^1 on $R^{n*} \odot R^{n*}$ is introduced as follows. If $g \in R^{n*} \odot R^{n*}$, $g = (g_{ij})$, and $j_0^1\alpha \in L_n^1$, $j_0^1\alpha = (a_i^p)$, then

$$j_0^1\alpha g = (a_p^i a_q^j g_{ij})$$

or more precisely, in our coordinates,

$$g_{pq}(j_0^1\alpha g) = a_p^i(j_0^1\alpha)a_q^j(j_0^1\alpha)g_{ij}(g) \tag{3.3}$$

After having introduced the coordinates and the group action (3.3) of L_n^1 on $R^{n*} \odot R^{n*}$ we are able to write the action (2.4) of L_n^3 on $T_n^2(R^{n*} \odot R^{n*})$ in terms of our coordinates. According to the general rules for computation with jets we construct a local map Φ belonging to the 2-jet $j_0^2(\tilde{\alpha}f\alpha^{-1})$, where $j_0^3\alpha \in L_n^3$, $j_0^2f \in T_n^2(R^{n*} \odot R^{n*})$. Putting

$$\Phi(x) = j_0^1(t_x\alpha t_{-\alpha^{-1}(x)})f(\alpha^{-1}(x))$$

we obtain, according to (3.2),

$$g_{pq}(\Phi(x)) = D_p(t_x\alpha t_{-\alpha^{-1}(x)})_i^{-1}(0)D_q(t_x\alpha t_{-\alpha^{-1}(x)})_j^{-1}(0)g_{ij}(f\alpha^{-1}(x))$$

Since for a suitable $y \in R^n$, $y = (y_1, y_2, \ldots, y_n)$,

$$(t_x \alpha t_{-\alpha^{-1}(x)})_i^{-1}(y) = -\alpha_i^{-1}(x) + \alpha_i^{-1}(x + y)$$

then, differentiating with respect to y_p,

$$D_p(t_x \alpha t_{-\alpha^{-1}(x)})_i^{-1}(0) = D_p \alpha_i^{-1}(0)$$

and

$$g_{pq}(\phi(x)) = D_p\alpha_i^{-1}(x) D_q \alpha_j^{-1}(x) g_{ij}(f\alpha^{-1}(x)) \tag{3.4}$$

The desired coordinate expression for the map (2.4) now follows from (3.2) and (3.4). Writing $j_0{}^2\Phi = j_0{}^3\alpha j_0{}^2 f$ and

$$g'_{pq} = g_{pq}(j_0{}^2\Phi) \qquad g'_{pq,r} = g_{pq,r}(j_0{}^2\Phi) \qquad g'_{pq,rs} = g_{pq,rs}(j_0{}^2\Phi)$$

we obtain after necessary differentiations that the following assertion holds.

Proposition. In terms of the coordinates (3.1) and (3.2), the natural action of $L_n{}^3$ on $T_n{}^2(R^{n*} \odot R^{n*})$ is given by

$$\begin{aligned} g'_{pq} &= a_p{}^i a_q{}^j g_{ij} \\ g'_{pq,r} &= a_p{}^i a_q{}^j a_r{}^k g_{ij,k} + (a^i_{rp} a_q{}^j + a_p{}^i a^j_{rq}) g_{ij} \\ g'_{pq,rs} &= a_p{}^i a_q{}^j a_r{}^k a_s{}^l g_{ij,kl} \\ &\quad + (a_p{}^i a_q{}^j a^k_{rs} + a^i_{ps} a_q{}^j a_r{}^k + a_p{}^i a^j_{sq} a_r{}^k + a^i_{rp} a_q{}^j a_s{}^k + a_p{}^i a^j_{rq} a_s{}^k) g_{ij,k} \\ &\quad + (a^i_{srp} a_q{}^j + a^i_{rp} a^j_{sq} + a^i_{sp} a^j_{rq} + a_p{}^i a^j_{srq}) g_{ij} \end{aligned} \tag{3.5}$$

A point $j_0{}^2 f \in T_n{}^2(R^{n*} \odot R^{n*})$, $j_0{}^2 f = (g_{ij}, g_{ij,k}, g_{ij,kl})$, will be called regular if

$$\det(g_{ij}) \neq 0 \tag{3.6}$$

Denote by $\delta_k{}^i$ the Kronecker symbol and by g^{ij} the functions of the coordinates g_{ij} introduced on a neighborhood of a regular point $j_0{}^2 f \in T_n{}^2(R^{n*} \odot R^{n*})$ by

$$g^{ik} g_{kl} = \delta_l{}^i$$

To investigate the map (2.4) further, we shall simplify the formulas (3.5) by using a new system of local coordinates on this neighborhood. The following assertion can be proved by a direct calculation.

Proposition. The formulas

$$\begin{aligned} g_{ij} &= g_{ij} \\ \Gamma_{i,jk} &= \tfrac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i}) \\ R_{ijkl} &= \tfrac{1}{2}(g_{il,jk} + g_{jk,il} - g_{ik,jl} - g_{jl,ik}) \\ &\quad + \tfrac{1}{4} g^{mp}((g_{mj,k} + g_{mk,j} - g_{jk,m})(g_{pi,l} + g_{pl,i} - g_{il,p}) \\ &\quad - (g_{mj,l} + g_{ml,j} - g_{jl,m})(g_{pi,k} + g_{pk,i} - g_{ik,p})) \\ S_{i,jkl} &= \tfrac{1}{3}(g_{ij,kl} + g_{il,jk} + g_{ik,lj}) - \tfrac{1}{6}(g_{jk,li} + g_{lj,ki} + g_{kl,ji}) \end{aligned} \tag{3.7}$$

define a coordinate transformation on a neighborhood of a regular point $j_0^2 f \in T_n^2(R^{n*} \odot R^{n*})$. The inverse transformation is given by

$$
\begin{aligned}
g_{ij} &= g_{ij} \\
g_{ij,k} &= \Gamma_{i,jk} + \Gamma_{j,ik} \\
g_{ij,kl} &= S_{i,jkl} + S_{j,ikl} - \tfrac{1}{3}(R_{ikjl} + R_{jkil}) \\
&\quad + \tfrac{1}{3} g^{pq}(\Gamma_{p,il}\Gamma_{q,kj} + \Gamma_{p,jl}\Gamma_{q,ki} - 2\Gamma_{p,ij}\Gamma_{q,kl})
\end{aligned}
$$

Notice that the coordinates R_{ijkl} on $T_n^2(R^{n*} \odot R^{n*})$ satisfy the algebraic identities

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij} \qquad R_{ijkl} + R_{iljk} + R_{iklj} = 0 \tag{3.8}$$

Owing to these identities, there are $\frac{1}{12}n^2(n^2 - 1)$ independent coordinates R_{ijkl} (Eisenhart, 1964, p. 21).

Let us express the map (2.4) in terms of the coordinates g_{ij}, $\Gamma_{i,jk}$, R_{ijkl}, $S_{i,jkl}$ (3.7).

Proposition. On a neighborhood of any regular point

$$j_0^2 f \in T_n^2(R^{n*} \odot R^{n*}) \qquad j_0^2 f = (g_{ij}, g_{ij,k}, g_{ij,kl})$$

the map (2.4) is expressed by

$$
\begin{aligned}
g'_{pq} &= a_p{}^i a_q{}^j g_{ij} \\
\Gamma'_{p,qr} &= a_p{}^i a_q{}^j a_r{}^k \Gamma_{i,jk} + a_p{}^i a^j_{rq} g_{ij} \\
R'_{pqrs} &= a_p{}^i a_q{}^j a_r{}^k a_s{}^l R_{ijkl} \\
S'_{p,qrs} &= a_p{}^i a_q{}^j a_r{}^k a_s{}^l S_{i,jkl} \\
&\quad + [a_p{}^i(a_q{}^j a^k_{rs} + a_s{}^j a^k_{rq} + a_r{}^j a^k_{qs}) + a_p{}^j(a^i_{rs} a_q{}^k + a^i_{rq} a_s{}^k + a^i_{sq} a_r{}^k) \\
&\quad + \tfrac{1}{3}(a^i_{ps} a_r{}^j a_q{}^k + a^i_{pr} a_s{}^j a_q{}^k + a^i_{pq} a_r{}^j a_s{}^k)] \\
&\quad \times \Gamma_{i,jk} + a_p{}^i[a^j_{qrs} + \tfrac{1}{3}(a^i_{rp} a^j_{sq} + a^i_{sp} a^j_{rq} + a^i_{pq} a^j_{rs})] g_{ij}
\end{aligned}
\tag{3.9}
$$

We are now in a position to express the Lie algebra homomorphism (2.2) in terms of our coordinates. Denote by *id* the identity map of R^n, and by e the identity element of the group L_n^3:

$$e = j_0^3 id = (\delta_j{}^i, 0, 0) \tag{3.10}$$

Let $\xi \in T_e L_n^3$ be any element of the Lie algebra l_n^3. By definition of the tangent map,

$$T_e \tilde{Q} \xi = (\tilde{Q}(e), D\tilde{Q}(e)\xi) = (Q, D\tilde{Q}(e)\xi)$$

Let $e_p{}^i$, e_p^{ij}, e_p^{ijk} be the basis of $T_eL_n{}^3$ relative to the coordinates $a_i{}^p$, a_{ij}^p, a_{ijk}^k. Then

$$\xi = \xi_i{}^p e_p{}^i + \xi_{ij}^p e_p^{ij} + \xi_{ijk}^p e_i^{pjk}$$

where the components ξ_{ij}^p, ξ_{ijk}^p are symmetric in the subscripts. Then by (3.9),

$$T_e\tilde{Q}\xi = Dg'_{pq}(e)\xi\frac{\partial}{\partial g_{pq}} + D\Gamma'_{p,qr}(e)\xi\frac{\partial}{\partial\Gamma_{p,qr}}$$

$$+ DR'_{pqrs}(e)\xi\frac{\partial}{\partial R_{pqrs}} + DS'_{p,qrs}(e)\xi\frac{\partial}{\partial S_{p,qrs}}$$

where $j_0{}^2f \in T_n{}^2(R^{n*} \odot R^{n*})$ is fixed, and the partial derivatives on the right denote the tangent vectors to $T_n{}^2(R^{n*} \odot R^{n*})$ at $j_0{}^2f$ associated with the coordinates (3.2). Putting

$$T_e\tilde{Q}\xi = \xi_j{}^i\Xi_i{}^j + \xi_{jk}^i\Xi_i^{jk} + \xi_{jkl}^i\Xi_i^{jkl}$$

we easily obtain

$$\Xi_i{}^j = \frac{\partial g'_{pq}}{\partial a_j{}^i}\frac{\partial}{\partial g_{pq}} + \frac{\partial\Gamma'_{p,qr}}{\partial a_j{}^i}\frac{\partial}{\partial\Gamma_{p,qr}} + \frac{\partial R'_{pqrs}}{\partial a_j{}^i}\frac{\partial}{\partial R_{pqrs}} + \frac{\partial S'_{p,qrs}}{\partial a_j{}^i}\frac{\partial}{\partial S_{p,qrs}},$$

$$\Xi_i^{jk} = \frac{\partial\Gamma'_{p,qr}}{\partial a_{jk}^i}\frac{\partial}{\partial\Gamma_{p,qr}} + \frac{\partial S'_{p,qrs}}{\partial a_{jk}^i}\frac{\partial}{\partial S_{p,qrs}}$$

$$\Xi_i^{jkl} = \frac{\partial S'_{p,qrs}}{\partial a_{jkl}^i}\frac{\partial}{\partial S_{p,qrs}}$$

where the partial derivatives are considered at the point e. A direct computation gives

$$\Xi_i^{jkl} = g_{im}\frac{\partial}{\partial S_{m,jkl}}$$

We set

$$\Xi^{i,jkl} = g^{im}\Xi_m^{jkl} = \frac{\partial}{\partial S_{i,jkl}}$$

Further,

$$\frac{\partial\Gamma'_{p,qr}}{\partial a_{jk}^i}\frac{\partial}{\partial\Gamma_{p,qr}} = g_{ip}\frac{\partial}{\partial\Gamma_{p,jk}}$$

and we set

$$\Xi^{i,jk} = g^{im}\left(\Xi_m^{jk} - \frac{\partial S'_{p,qrs}}{\partial a_{jk}^m}\Xi^{p,qrs}\right) = \frac{\partial}{\partial\Gamma_{i,jk}}$$

Similarly,

$$\frac{\partial g'_{pq}}{\partial a_j^i}\frac{\partial}{\partial g_{pq}} = 2g_{im}\frac{\partial}{\partial g_{jm}} \qquad \frac{\partial R'_{pqrs}}{\partial a_j^i}\frac{\partial}{\partial R_{pqrs}} = 4R_{ipqr}\frac{\partial}{\partial R_{jpqr}}$$

and we set,

$$\Xi^{i,j} = \tfrac{1}{2}g^{im}\left(\Xi_m{}^j - \frac{\partial \Gamma'_{p,qr}}{\partial a_j{}^m}\Xi^{p,qr} - \frac{\partial S'_{p,qrs}}{\partial a_j{}^m}\Xi^{p,qrs}\right) = \frac{\partial}{\partial g_{ij}} + 2g^{im}R_{mpqr}\frac{\partial}{\partial R_{jpqr}}$$

It is clear that each of the systems $\Xi_i{}^j$, Ξ_i^{jk}, Ξ_i^{jkl} and $\Xi^{i,j}$, $\Xi^{i,jk}$, $\Xi^{i,jkl}$ of fundamental vector fields span the Lie algebra $l_{L_n{}^3}(T_n{}^2(R^{n*} \odot R^{n*}))$ around any regular point.

It is convenient to express our vector fields in the coordinates g^{ij} instead of g_{ij}. Putting

$$\tilde{\Xi}_{i,j} = -g_{ik}g_{jl}\Xi^{k,l} = \frac{\partial}{\partial g^{ij}} - 2g_{ip}R_{jqrs}\frac{\partial}{\partial R_{pqrs}}$$

and

$$\Xi_{ij}^+ = \tfrac{1}{2}(\tilde{\Xi}_{i,j} + \tilde{\Xi}_{j,i}) \qquad \Xi_{ij}^- = \tfrac{1}{2}(\tilde{\Xi}_{i,j} - \tilde{\Xi}_{j,i})$$

we have proved the following result.

> *Theorem.* Around any regular point of the manifold $T_n{}^2(R^{n*} \odot R^{n*})$, the Lie algebra $l_{L_n{}^3}(T_n{}^2(R^{n*} \odot R^{n*}))$ is spanned by the vector fields $\Xi^{i,jkl}$, $\Xi^{i,jk}$, Ξ_{ij}^+, Ξ_{ij}^-.

In general, the vector fields $\Xi^{i,jkl}$, $\Xi^{i,jk}$, Ξ_{ij}^+, and Ξ_{ij}^- do not have to be linearly independent. Consequently, the dimension of the Lie algebra $l_{L_n{}^3}(T_n{}^2(R^{n*} \odot R^{n*}))$ on the considered open sets is less than or equal to

$$N = n\left[n + \binom{n+1}{2} + \binom{n+2}{3}\right]$$

Recall that an $L_n{}^3$-invariant function defined on an open $L_n{}^3$-invariant subset of $T_n{}^2(R^{n*} \odot R^{n*})$ is called generally invariant. Our foregoing discussion may be summarized as follows.

> *Theorem.* Each generally invariant function L, defined on an open $L_n{}^3$-invariant subset of regular points of the manifold $T_n{}^2(R^{n*} \odot R^{n*})$, satisfies the system
>
> $$\Xi^{i,jkl}(L) = 0 \qquad \Xi^{i,jk}(L) = 0 \qquad \Xi_{ij}^+(L) = 0 \qquad \Xi_{ij}^-(L) = 0 \qquad (3.11)$$
>
> of partial differential equations. In particular, each generally invariant function depends only on g^{ij} (or g_{ij}) and R_{ijkl}.

The relations (3.11) may be called a complete system of differential identities for the considered class of generally invariant Lagrangians.

All our considerations can be repeated in the coordinates g_{ij}, $g_{ij,k}$, $g_{ij,kl}$ (3.2). Using (3.7) instead of (3.9) we obtain the following result.

Proposition. Around any regular point of the manifold $T_n^2(R^{n*} \odot R^{n*})$, the Lie algebra $l_{L_n^3}(T_n^2(R^{n*} \odot R^{n*}))$ is spanned by the vector fields

$$\theta_i^j = 2g_{im} \frac{\partial}{\partial g_{jm}} + (g_{pq,i}\delta_r^j + 2g_{iq,r}\delta_p^j) \frac{\partial}{\partial g_{pq,r}}$$

$$+ 2(g_{pq,ri}\delta_s^j + g_{iq,rs}\delta_p^j) \frac{\partial}{\partial g_{pq,rs}}$$

$$\theta_i^{jk} = g_{iq}(\delta_r^j\delta_p^k + \delta_p^j\delta_r^k) \frac{\partial}{\partial g_{pq,r}}$$

$$+ [g_{pq,i}\delta_r^j\delta_s^k + 2g_{iq,r}(\delta_p^j\delta_s^k + \delta_s^j\delta_p^k)] \frac{\partial}{\partial g_{pq,rs}}$$

$$\theta_i^{jkl} = \tfrac{2}{3}g_{iq}(\delta_s^j\delta_r^k\delta_p^l + \delta_s^j\delta_p^k\delta_r^l + \delta_p^j\delta_s^k\delta_r^l) \frac{\partial}{\partial g_{pq,rs}}$$

Theorem. Each L_n^3-invariant function L, defined on an open L_n^3-invariant subset of regular points of the manifold $T_n^2(R^{n*} \odot R^{n*})$, satisfies the conditions

$$\theta_i^j(L) = 0 \qquad \theta_i^{jk}(L) = 0 \qquad \theta_i^{jkl}(L) = 0 \tag{3.12}$$

4. REMARKS

Let us characterize the Lagrangians L as depending only on g_{ij} and $g_{ij,k}$, not on $g_{ij,kl}$. For these Lagrangians, equations (3.12) are equivalent with the system

$$\tilde{\theta}_i^j(L) = 0 \qquad \tilde{\theta}_i^{jk}(L) = 0$$

where

$$\tilde{\theta}_i^j = 2g_{im} \frac{\partial}{\partial g_{jm}} + (g_{pq,i}\delta_r^j + 2g_{iq,r}\delta_p^j) \frac{\partial}{\partial g_{pq,r}}$$

$$\tilde{\theta}_i^{jk} = g_{iq}(\delta_r^j\delta_p^k + \delta_p^j\delta_r^k) \frac{\partial}{\partial g_{pq,r}}$$

It is easily seen that around any regular point $(g_{ij}, g_{ij,k})$

$$\frac{\partial}{\partial g_{kl,j}} = \tfrac{1}{2}(g^{ml}\tilde{\theta}_m^{jk} + g^{mk}\tilde{\theta}_m^{jl} - g^{mj}\tilde{\theta}_m^{lk})$$

It is now evident that each generally invariant Lagrangian L satisfies

$$\frac{\partial L}{\partial g_{kl,j}} = 0 \qquad \frac{\partial L}{\partial g_{pq}} = 0$$

This is a proof of the well-known fact that there are no nontrivial generally invariant Lagrangians that depend on the components of a metric field and their first derivatives only.

Our second remark is intended to illustrate the notion of the flat local coordinate system (Hermann, 1968), and the geometrical meaning of the "generally invariant functions" as some coordinates belonging to a flat local coordinate system.

Let us consider the vector-field system $\Xi_{ij}^{+}, \Xi_{ij}^{-}, \Xi^{i,jk}, \Xi^{i,jkl}$ on $T_2{}^2(R^{2*} \odot R^{2*})$. In this case, there is only one independent variable among the functions R_{ijkl}, say R_{1212}. It is immediately seen that $\Xi_{ij}^{-} = 0$, $i, j = 1, 2$. Put $\tilde{g} = \det(g^{ij})$ and introduce a coordinate transformation from g^{11}, g^{12}, g^{22}, R_{1212} to $g^{11}, g^{12}, g^{22}, R$ by

$$g^{ij} = g^{ij} \qquad R = g^{ik}g^{jl}R_{ijkl} = 2\tilde{g}R_{1212}$$

One immediately obtains

$$\Xi_{ij}^{+} = \frac{\partial}{\partial g_{ij}} + 2g^{pq}R_{piqj}\frac{\partial}{\partial R} - (g_{ip}R_{jqrs} + g_{jp}R_{iqrs})g^{pr}g^{qs}\frac{\partial}{\partial R} = \frac{\partial}{\partial g_{ij}}$$

Summarizing the discussion of the case $n = 2$ we see that around each regular point the Lie algebra $l_{L_2{}^3}(T_2{}^2(R^{2*} \odot R^{2*}))$ is spanned by the vector fields $\partial/\partial S_{i,jkl}$, $\partial/\partial \Gamma_{i,jk}$, $\partial/\partial g^{ij}$. The coordinates $S_{i,jkl}$, $\Gamma_{i,jk}$, g^{ij}, R establish a flat local coordinate system for the vector-field system $\Xi_{ij}^{+}, \Xi_{ij}^{-}, \Xi^{ij,k}, \Xi^{i,jkl}$. In particular, there is only one independent generally invariant function R.

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